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# Coherent states for quons

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**Abstract.** Coherent states for quons are constructed as eigenstates of the annihilation operator and their properties are studied. In particular, it is shown that these states form a complete set with respect to a measure.

## 1. Introduction

Parastatistics was introduced by Green (1953) who observed that it is not necessary for all particles in nature to be either bosons or fermions because the principles of quantum field theory allow a more general statistics which includes the usual ones as limiting cases. More recently Greenberg (1991, 1992) and Mahapatra (1990) have considered the possible small violation of the Pauli principle as an experimental test for confirming the existence of such statistics. These authors introduce a deformed algebra containing a parameter $q$ $(-1 \leqslant q \leqslant 1)$. For $q = -1$ or $+1$, the algebra reduces to the fundamental anticommutation or commutation relations appropriate to fermions or bosons, respectively. The intermediate statistics for other values of $q$ smoothly interpolates between the two observed limits. The elementary excitations of such fields were named *quons*. The Fock-like space generated by the quonic excitation operator is a positively normed space and a unique number operator with integer eigenvalues can be constructed. The case $q = 0$ is somewhat special being a singular limit (Mahapatra 1990) for the number operator. It has been studied in greater detail (Greenberg 1990).

In this paper we confine ourselves mainly to the interval $0 \leqslant q \leqslant 1$. The case $q < 0$ will be briefly touched upon in our concluding remarks. Since $q = 1$ is the bosonic case and the harmonic oscillator coherent states or Glauber coherent states are well known and extensively studied (Glauber 1963, Sudarshan 1963) we construct the quonic coherent states appropriate to the above-mentioned interval and show that it has properties consistent with the bosonic limit. The question of completeness is dealt with in some detail through an explicit construction of the measure. The paper is organized as follows. We start with a brief review of the quonic algebra in section 2. Sections 3 and 4 constitute the main body of the paper. We conclude in section 5 with a few pertinent remarks.

## 2. The quon algebra

Let us introduce the quon creation operator $a^\dagger$ and the corresponding annihilation operator $a$ for a single level. They constitute a deformed algebra with the following relation (appropriately termed a $q$-mutator),

$$aa^\dagger - qa^\dagger a = 1 \qquad -1 \leqslant q \leqslant 1. \tag{1}$$

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We assume the existence of a unique vacuum $|0\rangle$ which is destroyed by the annihilation operator

$$a|0\rangle = 0. \tag{2}$$

As usual the number states are generated by repeated applications of the creation operator on the vacuum,

$$\begin{aligned} a^{\dagger}|0\rangle &= t_1|1\rangle \qquad t_1 \equiv 1 \\ a^{\dagger}|1\rangle &= t_2|2\rangle \\ &\vdots \\ a^{\dagger}|n-1\rangle &= t_n|n\rangle \end{aligned} \tag{3}$$

where the coefficients $t_i$ are taken for simplicity to be real. We may, therefore, write

$$\begin{aligned} |n\rangle &= (t_1 t_2 \ldots t_n)^{-1} (a^{\dagger})^n |0\rangle \\ &\equiv \frac{(a^{\dagger})^n}{h_n}|0\rangle. \end{aligned} \tag{4}$$

The matrix elements of $a$ and $a^{\dagger}$ in the number states are therefore of the form

$$\langle m|a|n\rangle = t_n \delta_{m,n-1} \qquad \langle m|a^{\dagger}|n\rangle = t_{n+1}\delta_{m,n+1}. \tag{5}$$

Taking matrix elements of (1) and using (5), we arrive at a recursion relation for the coefficients $t_n$, namely

$${t_n}^2 - q{t_{n-1}}^2 = 1 \tag{6}$$

which is readily solved to yield

$${t_n}^2 = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}. \tag{7}$$

For bosons ($q = 1$), $t_n = \sqrt{n}$ as expected. For fermions, on the other hand, $t_2 = 0$ and only the vacuum and one particle state survive to account for the Pauli principle. For $q = 0$ (infinite statistics), all the $t_n$ become 1.

A number operator for $q > 0$ has been devised (Mahapatra 1990) and takes the form

$$N = \frac{\ln[1-(1-q)N_0]}{\ln(q)} \tag{8}$$

where $N_0 = a^{\dagger}a$. Writing $q = 1 - \epsilon$, it is easy to see that $N|n\rangle = n|n\rangle$. For $q = 1$, $N$ is simply $a^{\dagger}a$ as it should be. However, for $q = 0$, $N$ is singular. In this case it is easy to check that the appropriate number operator (Mahapatra 1990) is

$$N = \sum_{m=1}^{\infty} (a^{\dagger})^m (a)^m. \tag{9}$$

The states $|n\rangle$ as eigenstates of a Hermition number operator $N$ provide a complete orthonormal basis,

$$\sum_n |n\rangle\langle n| = \mathbf{1} \qquad \langle n|m\rangle = \delta_{n,m}. \tag{10}$$

## 3. The coherent state

The most general way to introduce a coherent state is through the action of a unitary displacement operator on a reference state (Perelomov 1986, Zhang *et al* 1990) which may be the vacuum. Other methods involve the construction of an eigenstate of the annihilation operator (Glauber 1963, Sudarshan 1963) or a state which yields a minimum uncertainty product (Nieto and Simmons 1978). Depending on the nature of the spectrum, one or the other of the above methods may be employed. In the case of the harmonic oscillator, when the vacuum is the reference state, all three approaches result in the same state.

In the present case we define the coherent state as an eigenstate of the annihilation operator, namely

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{11}$$

where the eigenvalue $\alpha$ is a complex number. We substitute the expansion of $|\alpha\rangle$ in the number states,

$$|\alpha\rangle = \sum_n c_n|n\rangle \tag{12}$$

in equation (11) and equate the coefficients of $|n\rangle$. This yields

$$c_n = c_0\frac{\alpha^n}{h_n} \qquad h_0 \equiv 1. \tag{13}$$

$c_0$ is obtained by imposing the normalization condition $\langle\alpha|\alpha\rangle = 1$ and the normalized coherent state takes the form

$$|\alpha\rangle = \left[\sum_n \frac{|\alpha|^{2n}}{{h_n}^2}\right]^{-1/2} \sum_{n=o}^{\infty} \frac{\alpha^n}{h_n}|n\rangle. \tag{14}$$

We define a function $f(s)$ as

$$f(s) = \sum_{n=o}^{\infty} \frac{s^n}{{h_n}^2} \tag{15}$$

and use (4) to arrive at the following convenient form for the normalized coherent state:

$$|\alpha\rangle = \left[f(|\alpha|^2)\right]^{-1/2} f(\alpha a^\dagger)|0\rangle. \tag{16}$$

The bosonic limit is reproduced by observing that for $q = 1$, $h_n = \sqrt{n!}$ and consequently $f(s) = \mathrm{e}^s$.

The average number of particles in the state $|\alpha\rangle$ is given by the diagonal matrix element of the number operator defined in (8). It is easily found to be

$$\langle\alpha|N|\alpha\rangle = |\alpha|^2 \frac{\mathrm{d}}{\mathrm{d}(|\alpha|^2)} \ln f(|\alpha|^2) \tag{17}$$

and reduces to $|\alpha|^2$ in the bosonic case as expected.

The function $f(s)$ as given by (15) is convergent within a circle whose radius increases from 1 for $q = 0$ to infinity for $q = 1$. Outside this circle the function will be defined by analytic continuation. It is easy to find an expression for the radius of convergence. The ratio of the $(n+1)$th to the $n$th term of the series for $f(s)$ is $s(1-q)/(1-q^{n+1})$ which goes to $s(1-q)$ as $n$ approaches $\infty$. Hence the radius of convergence is $1/(1-q)$. Since the departure of the function $\ln f(s)$ from the diagonal line ($\ln f(s) = s$) indicates

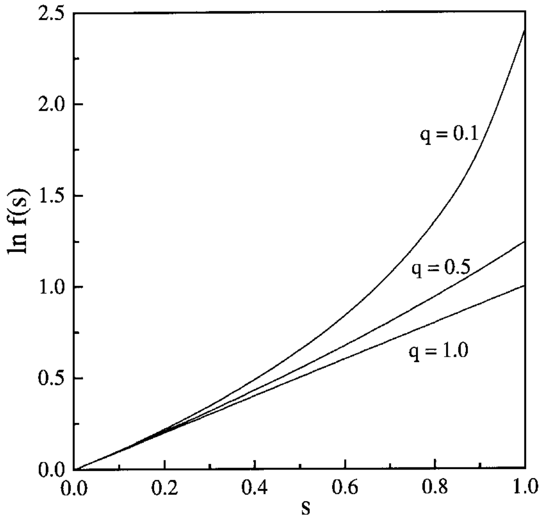


**Figure 1.** The function $\ln f(s)$ is displayed as a function of $s$. The straight line corresponds to $q = 1$.

the departure from the bosonic limit, we give a plot (figure 1) of the function for several values of $q$. On the other hand, for $q = 0$ we have to use expression (9) for the number operator to obtain

$$\langle\alpha|N|\alpha\rangle = \frac{|\alpha|^2}{1-|\alpha|^2}. \tag{18}$$

In this case, only states with $|\alpha|^2 < 1$ will be allowed.

The inner product of two coherent states $|\alpha\rangle$ and $|\beta\rangle$ can be readily found as

$$\langle\alpha|\beta\rangle = \frac{f(\alpha^*\beta)}{[f(|\alpha|^2)f(|\beta|^2)]^{1/2}} \tag{19}$$

showing that the states are not orthogonal. This is expected because the operator $a$ is not Hermitian.

To investigate the time evolution of these coherent states let us consider a simple Hamiltonian such as $H = \omega N$. Since the time evolution of each stationary state is given by $|n(t)\rangle = |n(0)\rangle \exp(-\mathrm{i}n\omega t)$ it is easy to see that under this Hamiltonian a coherent state will remain a coherent state with $\alpha(t) = \alpha(0)\exp(-\mathrm{i}\omega t)$. This is also evident from the Heisenberg equation of motion for $a$, namely $\mathrm{d}a/\mathrm{d}t = -\mathrm{i}[a, H] = -\mathrm{i}\omega a$, since the operators $[a, N]$ and $a$ are equal, both having the same matrix elements.

## 4. Completeness of the coherent states

We now investigate the completeness property of the coherent states. We assume the existence of a weight function $\mu(\alpha)$ such that the resolution of the identity operator reads

$$\int |\alpha\rangle\langle\alpha|\mu(\alpha)\,\mathrm{d}^2\alpha = 1 \tag{20}$$

where $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\mathrm{d}(\mathrm{Im}\,\alpha)$. Substituting for $|\alpha\rangle$ in (20), we have

$$\sum_{m,n}\int [f(|\alpha|^2)]^{-1}\frac{\alpha^{*m}\alpha^n}{h_m h_n}|n\rangle\langle m|\mu(\alpha)\,\mathrm{d}^2\alpha = 1. \tag{21}$$

Taking diagonal matrix elements and using the orthonormality of the $|n\rangle$ states, we see that

$$\int [f(|\alpha|^2)]^{-1}(|\alpha|^2)^n \mu(\alpha)\,\mathrm{d}^2\alpha = h_n^2. \tag{22}$$

On the other hand, writing $\mathrm{d}^2\alpha = \frac{1}{2}\mathrm{d}(|\alpha|^2)\,\mathrm{d}\phi$ and $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\phi}$, we can use

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)\phi}\,\mathrm{d}\phi = \delta_{n,m}$$

only when $\mu(\alpha)$ is independent of the phase of $\alpha$, i.e. $\mu(\alpha) = \mu(|\alpha|^2)$ and arrive at (22). We, therefore, write

$$\int_0^\infty [f(s)]^{-1} s^n \mu(s)\,\mathrm{d}s = \frac{h_n^2}{\pi} \tag{23}$$

where $s$ stands for $|\alpha|^2$.

Equation (23) provides the moments of $\mu(s)/f(s)$. The problem of finding a suitable measure thus reduces to a moment problem (Mukunda *et al* 1980, Sharma *et al* 1981).

By multiplying both sides of (23) by $(\mathrm{i}y)^n/n!$ and summing over $n$ we obtain

$$\int_0^\infty [f(s)]^{-1}\mathrm{e}^{\mathrm{i}sy}\mu(s)\,\mathrm{d}s = \phi(y). \tag{24}$$

The function $\phi(y)$, given by

$$\phi(y) = \frac{1}{\pi}\sum_{n=0}^\infty \frac{(\mathrm{i}y)^n h_n^2}{n!}$$

is absolutely convergent for $|y| < 1$ and for $|y| > 1$ may be defined by analytic continuation. We can now determine the weight function by taking an inverse Fourier transformation of equation (24). The result is

$$\mu(s) = \frac{f(s)}{2\pi}\int_{-\infty}^\infty \phi(y)\mathrm{e}^{-\mathrm{i}sy}\,\mathrm{d}y. \tag{25}$$

We readily check that (25) yields the right measure for the bosonic coherent states for which $\phi(y) = 1/\pi(1-\mathrm{i}y)$. This has a pole at $y = -\mathrm{i}$ and an integration in the lower-half plane enclosing the pole yields $\mu(s) = 1/\pi$. However, for infinite statistics the measure is singular. In this case, $\phi(y) = \mathrm{e}^{\mathrm{i}y}/\pi$ and consequently $\mu(s) = 2f(s)\delta(s-1)$ which shows that the coherent states for $q = 0$ are complete on the unit circle. However, these are the states for which the average number of particles is infinite.

By virtue of (20) an arbitrary state $|\beta\rangle$ has an expansion in the coherent state basis, namely

$$|\beta\rangle = \int \frac{f(\alpha^*\beta)}{[f(|\alpha|^2)f(|\beta|^2)]^{1/2}}|\alpha\rangle\mu(\alpha)\,\mathrm{d}^2\alpha. \tag{26}$$

The above equation shows that the coherent states introduced form an overcomplete set (Perelomov 1986). Moreover from (26) we also observe the self-reproducing property of the $f$'s, namely

$$(\alpha^*\beta) = \int \frac{f(\alpha^*\gamma)f(\gamma^*\beta)}{f(|\gamma|^2)}\mu(\gamma)\,\mathrm{d}^2\gamma. \tag{27}$$

## 5. Concluding remarks

So far we have not said anything about the $q < 0$ case. We observe that the construction given by (14) and (16) is valid for parafermions as well. The only difference is that the radius of convergence of the series for $f(s)$ is further reduced to $1/(1+|q|)$. The impossibility of annihilation operator coherent states for a finite spectrum is reflected in the singular nature of the fermionic limit where all the $h_n$'s are zero except $h_0$ and $h_1$ which are 1. However, since states higher than $|1\rangle$ are absent expression (14) yields

$$|\alpha\rangle = [1+|\alpha|^2]^{-1/2}[|0\rangle + \alpha|1\rangle]$$

which can be identified with the angular momentum coherent states for $j = \frac{1}{2}$ obtained by the action of the unitary displacement operator $\exp(\beta a^\dagger - \beta^* a)$ on the state of lowest weight, namely $|\frac{1}{2}, -\frac{1}{2}\rangle$ where $\beta$ and $\alpha$ are related through the following parametrization: $\beta = (\theta/2)\exp(-\mathrm{i}\phi)$, $\alpha = \tan(\theta/2)\exp(-\mathrm{i}\phi)$ $(0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi)$. This is due to the fact that the fermionic operators $a, a^\dagger$ and $(a^\dagger a - \frac{1}{2})$ satisfy the same algebra as $J_-$, $J_+$ and $J_0$ for $j = \frac{1}{2}$. The proof for completeness goes through for $q < 1$ as well. The weight factor $\mu(s)$ in this case is non-zero within the unit circle. This can be inferred from (23) showing that the higher moments are damped out faster as $q$ approaches $-1$. Again this argument does not hold in the fermionic limit where the measure is $(1/2\pi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, as is appropriate for displacement operator coherent states. Further investigations into the nature of these coherent states are hampered by the fact that a suitable number operator is not available for $q < 0$.

In a recent paper Campos (1994) defines the coherent state as a displacement of the vacuum. He writes

$$|\alpha\rangle = \mathcal{N}\exp(\alpha a^\dagger)|0\rangle = \mathcal{N}\sum_n \frac{\alpha^n h_n}{n!}|n\rangle$$

where the normalization constant $\mathcal{N}$ is given by

$$\mathcal{N} = \left[\sum_n \frac{(|\alpha|)^{2n} h_n^2}{(n!)^2}\right]^{-1/2}.$$

This agrees with our coherent states only in the bosonic limit as it should. However, the problem of defining a coherent state for general $q$ by using a unitary operator, still remains. A related problem awaiting this particular development is the construction of squeezed coherent states which are obtained by successively applying a unitary squeezing operator first and then a unitary displacement operator on the vacuum.

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